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Modeling an accretion disc stochastical variability

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ABSTRACT
Hot spots residing on the surface of an accretion disc have been considered as a model of short-term variability of active galactic nuclei. In this paper we apply the theory of random point processes to model the observed signal from an ensemble of randomly generated spots. The influence of general relativistic effects near a black hole is taken into account and it is shown that typical features of power spectral density can be reproduced. Connection among spots is also discussed in terms of Hawkes' process, which produces more power at low frequencies. We derive a semi-analytical way to approximate the resulting power-spectral density.

Keywords: Black holes – Accretion – Variability

1 INTRODUCTION

Radiation from accreting black holes varies on different timescales [1]. In X-rays, the observed light-curve, $f \equiv f(t)$, is a complicated noisy curve that can be represented by a broad-band power spectrum [2]. It has been proposed [3, 4] that 'hot spots' are a possible contributor to this variability. These spots are supposed to occur on the surface of an accretion disc following its irradiation by coronal flares [5, 6, 7]. A model light-curve can be constructed as a sum of contributions from many point-like sources that are orbiting above an underlying accretion disc. The observed signal is modulated by relativistic effects as photons propagate towards a distant observer.

In order to characterise the light curves we need to introduce some appropriate estimator of the source variability. In a mathematical sense, one applies a functional: $f \to \mathcal{S}[f]$, where $\mathcal{S}[.]$ is a map from functions defined on $\mathbb{R}$ to functions on $\mathbb{R}^k$ ($k \geq 0$). The variability estimator can be a single number (for example, the mean flux or the 'rms' characteristic), or function of one variable (power spectrum density or probability distribution) or of many variables (poly-spectra, rms–flux relation, etc). A signal of such a spotted accretion disc should be intrinsically stochastical. Hence, the variability estimator $\mathcal{S}[f]$, derived from a piece of the light-curve, is a random value, too.

Various schemes have been proposed in which spots are mutually interconnected in some way [8, 9]. We want to investigate this type of models within a common mathematical basis. Other authors have developed different approaches to the problem [10, 11].

2 SPOT MODELS AND THE ACCRETION DISC VARIABILITY

2.1 Model assumptions and variables

Let us have K samples of the observed light-curves from the same source, f_j. The law of large numbers ensures that

$$\frac{1}{K}\sum_{j=1}^{K}\mathcal{S}\left[f_j\right]\to \mathrm{E}\left[\mathcal{S}\left[f\right]\right], \quad K\to\infty, \tag{1}$$

where E[.] is the mean value operator. The average value of the functional is formally defined

$$\mathrm{E}\left[\mathcal{S}\left[f\right]\right]\equiv \sum_{\{\text{All possible}\, f_j(t)\}} (\text{Probability of}\, f_j)\times\mathcal{S}\left[f_j\right], \tag{2}$$

where the sum goes over all possible light-curves generated by this model. We will show how to define and parameterise "the space of all possible light-curves" and how to perform the averaging when the functional is the power spectrum.

The general model is constrained only by the following three assumptions about the creation and evolution of spots:

(i) Each spot is described by its time and place of birth (t_j, r_j and ϕ_j) on the disc surface.
(ii) Every new occurrence starts instantaneously; afterwards the emissivity decays gradually to zero. The total emitted radiation energy is finite.
(iii) The intrinsic emissivity can be fully determined by a finite number of parameters which form a vector $\boldsymbol{\xi}_j$.

For a simple demonstration of this concept see figure 1. The disc itself has a passive role in our present considerations. We will treat it as a geometrically thin, optically thick layer lying in the equatorial plane.

2.2 Random point processes

The concept of point processes is a generalisation of well-known random processes which were developed as a description of time-dependent random values [12]. Point processes are used as statistical description of configurations of some randomly distributed points in space $\mathbb{R}^n$.

One way of describing a configuration of points is by their counting measure, $N(A)$, which for every $A \subset \mathbb{R}^n$ gives a number of points lying in A. One defines the intensity measure,

$$M_1(A) = \mathrm{E}\left[N(A)\right]. \tag{3}$$

Similarly to random processes, the point process can be characterised by its *mean value* and *moments*. For every $A \subset \mathbb{R}^n$, $M_1(A)$ is the mean number of points lying in A. The second-order moment measure is defined in the same way on the Cartesian product of spaces $\mathbb{R}^n \times \mathbb{R}^n$:

$$M_2(A \times B) = \mathrm{E}\left[N(A)N(B)\right]. \tag{4}$$

Let x_{iN} be one possible configuration of points, i.e. the support of some $N(.)$ For the functions $f(x)$ and $g(x,\, y)$ on $\mathbb{R}^n$ and $\mathbb{R}^{2n}$, respectively, it follows [13, 14]

$$\mathrm{E}\left[\sum_{\{x_i\}_N} f(x_i)\right] = \mathrm{E}\left[\int\limits_{\mathcal{X}} f(x)N(\mathrm{d}x)\right] = \int\limits_{\mathbb{R}^n} f(x)M_1(\mathrm{d}x) \tag{5}$$

$$\begin{aligned} \mathrm{E}\left[\sum_{\{x_i\}_N,\,\{y_i\}_N} g(x_i,\, y_i)\right] &= \mathrm{E}\left[\int\limits_{\mathbb{R}^{2n}} g(x,\, y)N(\mathrm{d}x)N(\mathrm{d}y)\right] \\ &= \int\limits_{\mathbb{R}^{2n}} g(x,\, y)M_2(\mathrm{d}x \times \mathrm{dy}). \end{aligned} \tag{6}$$

The concept of point process can be further generalised in the following way. We add a *mark* κ_i from the mark set $\mathcal{K}$ to each coordinate x_i from $\{x_i\}_N$. Marks carry additional information. The resulting point process on the set $\mathbb{R}^n \times \mathcal{K}$ is called the 'marked point process' if for every $A \subset \mathbb{R}^n$ it fulfills the condition $N_{\mathrm{g}}(A) \equiv N(A \times \mathcal{K}) < \infty$.

The random measure $N_{\mathrm{g}}(A)$ represents the *ground process* of the marked process N. When the dynamics of the process is governed only by the ground process and marks are mutually independent and identically distributed random values with the distribution functions $G(\mathrm{d}\kappa)$, then the process intensity and the second order measure fulfill

$$M_1(\mathrm{d}x \times \mathrm{d}\kappa) = M_{1\mathrm{g}}(\mathrm{d}x)G(\mathrm{d}\kappa), \tag{7}$$

$$M_2(\mathrm{d}x_1 \times \mathrm{d}\kappa_1 \times \mathrm{d}x_2 \times \mathrm{d}\kappa_2) = M_{\mathrm{g}2}(\mathrm{d}x_1 \times \mathrm{d}x_2)G(\mathrm{d}\kappa_1)G(\mathrm{d}\kappa_2). \tag{8}$$

2.3 Relationship between point processes and spots

Let us assume a surface element orbiting at radius r with constant emissivity I and orbital frequency $\Omega(r)$. This should represent an infinitesimally small spot. For the flux measured by an observer at inclination θ_{o} we find

$$f(t) = IF(t,\, r,\, \theta_{\mathrm{o}}). \tag{9}$$

The periodical modulation of the signal is determined by relations

$$F(t(\phi),\, r,\, \theta_{\rm o}) = F(\phi,\, r, \theta_{\rm o}), \tag{10}$$

$$t(\phi) = \frac{\phi}{\Omega(r)} + \delta t(\phi,\, r, \theta_{\rm o}), \tag{11}$$

where $F(\phi,\, r, \theta_{\rm o})$ is the transfer function describing the total amplification of signal emitted from then disc surface element on the coordinates r and ϕ. The function $\delta t(\phi,\, r, \theta_{\rm o})$ is the time delay of the signal (hereafter we will omit $\theta_{\rm o}$ in the argument of F for simplicity). Now, we consider a process consisting of statistically dependent events,

$$f(t) = \sum_j I(t - \delta_j,\, \boldsymbol{\xi}_j) F(t - \delta_{{\rm p}j},\, r_j), \tag{12}$$

where: $I(t,\, \boldsymbol{\xi}) = \theta(t) g(t,\, \boldsymbol{\xi})$ is the profile of a single event; $\delta_j = t_j + t_{0j}$ is time offset; $\delta_{{\rm p}j} = \delta_j + t_{{\rm p}j}$ is the phase offset; $\theta(t)$ is the Heaviside function; and $g(t,\, \boldsymbol{\xi})$ is non-negative function of $k+1$ variables t and $\boldsymbol{\xi} = (\xi^1, \ldots, \xi^k)$, which is on the interval $\langle 0,\, \infty)$ integrable in the variable t for all values of parameters $\boldsymbol{\xi} \in \Xi$. The set Ξ is some measurable subset of $\mathbb{R}^k$. For a fixed value of r, $F(t,\, r)$ is a periodical function of t, with the angular frequency $\Omega(r)$.

Quantities $\boldsymbol{\xi}_j$, t_j, r_j, $t_{{\rm p}j}$ and t_{0j} are random values. The vector $\boldsymbol{\xi}_j$ determines the duration and shape of each event, t_j is time of ignition of the j–th event, and t_{0j} the corresponding initial time-offset. Parameter $t_{{\rm p}j}$ determines the initial phase of the periodical modulation. Processes of this kind and their power spectra were mathematically studied by Brémaud and Massoulié, [15, 16].

Power spectral function of a stationary stochastic process $X(t)$ is

$$S(\omega) = \lim_{T \to \infty} \frac{1}{2T} {\rm E} \left[\left| \mathcal{F}_T[X(t)](\omega) \right|^2 \right], \tag{13}$$

where $\mathcal{F}_T[\,]$ is the incomplete Fourier transform,

$$\mathcal{F}_T[X(t)] = \int_{-T}^{T} X(t) e^{-i\omega t} {\rm d}t. \tag{14}$$

This can be evaluated by using the complete Fourier transform,

$$\int_{-T}^{T} X(t) e^{-i\omega t} {\rm d}t = \int_{-\infty}^{\infty} X(t) \chi_{\langle -T,\, T\rangle}(t) e^{-i\omega t} {\rm d}t = 2 \frac{\sin(T\omega)}{\omega} \star \mathcal{F}[X(t)](\omega), \tag{15}$$

where $\chi_A(x)$ is the characteristic function of set A, which equals 1 for $x \in A$ and 0 for $x \notin A$. Symbol $\star$ denotes the convolution operation.

By applying this transformation on the process (12) we find

$$\mathcal{F}_T[f(t)](\omega) = \frac{2\sin(T\omega)}{\omega} \star \sum_j \mathcal{F}[I(t - \delta_j,\, \boldsymbol{\xi}_j)\; F(t - \delta_{{\rm p}j},\, r_j)](\omega). \tag{16}$$

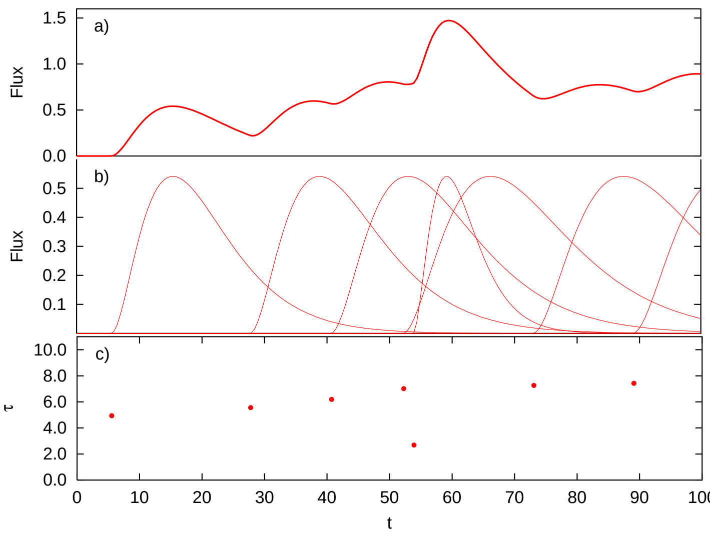

Figure 1. The model light-curve (panel a) is obtained as a sum of elementary events (panel b). Profile of each individual event is assumed to be $I(t, \tau) = (t/\tau)^2 \exp(-t/\tau)\,\theta(t)$, as described in the text. Their normalization is identical. The final light-curve is fully determined by the form of the individual contributions together with a set of points in t–τ plane (panel c), which represent pairs of ignition times and temporal constants τ of each event.

The Fourier transform of a single event $I(t-\delta_j,\, \boldsymbol{\xi}_j)F(t-\delta_{\mathrm{p}j},\, r_j)$ is then

$$\mathcal{F}[I(t-\delta_j,\, \boldsymbol{\xi}_j)F(t-\delta_{\mathrm{p}j},\, r_j)](\omega) = e^{-i\omega\delta_j}\mathcal{F}[I(t,\, \boldsymbol{\xi}_j)] \star \mathcal{F}[F(t+t_{\mathrm{p}j},\, r_j)]. \tag{17}$$

Function $F(t,\, r)$ is periodical in time, and so it can be expanded:

$$F(t,\, r) = \sum_{k=-\infty}^{\infty} c_k(r) e^{ik\Omega(r)t}, \tag{18}$$

where $\Omega(r)$ is the frequency of $F(t,\, r)$. We find

$$\mathcal{F}\left[F(t_{\mathrm{p}},\, r)\right](\omega) = \sum_{k=-\infty}^{\infty} c_k(r) e^{ik\Omega(r)t_{\mathrm{p}}}\,\delta\left(\omega - k\Omega(r)\right), \tag{19}$$

$$\mathcal{F}\left[I(t,\, \boldsymbol{\xi})\right] \star \mathcal{F}\left[F(t+t_{\mathrm{p}},\, r)\right] = \sum_{k=-\infty}^{\infty} c_k(r) e^{ik\Omega(r)t_{\mathrm{p}}}\mathcal{F}\left[I(t,\, \boldsymbol{\xi})\right]\left(\omega - k\Omega(r)\right). \tag{20}$$

The above given formulation of the problem falls perfectly within the mathematical framework of point processes.

2.4 The case of independent decaying spots (Poisson process)

Knowing the incomplete Fourier transform of $f(t)$ we can now calculate its squared absolute value and perform the averaging over all realizations of the process. Between $-T$ and T the process is influenced by all events ignited during the preceding interval $\langle -\infty, T\rangle$, however (because of fast decay of every single event), this can be restricted onto $\langle -(T+C), T\rangle$, where C is a sufficiently large positive constant. Therefore, every realization of the process $f(t)$ on the interval $\langle -T, T\rangle$ can be described by set of points in $(k+4)$–dimensional space $(t_j,\, t_{0j},\, t_{\mathrm{p}j},\, r_j,\, \boldsymbol{\xi}_j)$, where $t_j \in \langle -(T+C), T\rangle$.

Equation (12) represents a very general class of random processes. However, in all reasonable models of spotted accretion discs the values of initial time delay and phase are functions of initial position of each spot (r and ϕ), i.e.

$$t_0 = \delta t(r,\, \phi), \qquad t_{\mathrm{p}} = \frac{\phi}{\Omega(r)} + t_0. \tag{21}$$

Fourier transform of the resulting signal can be then simplified,

$$\begin{aligned}\mathcal{F}[I(t-t_{0j},\, \boldsymbol{\xi}_j) \ \ & F(t-t_{0j}+t_{\mathrm{p}j},\, r_j)](\omega) \\ &= \sum_{k=-\infty}^{\infty} c_k(r) e^{ik\phi} \mathcal{F}\left[I(t-\delta t(r,\, \phi),\, \boldsymbol{\xi})\right](\omega - k\Omega(r))\,.\end{aligned} \tag{22}$$

Every realization of this process is completely determined by set of points $(t_j,\, \phi_j,\, r_j,\, \boldsymbol{\xi}_j)$ from some subset of $\mathbb{R}^{k+3}$.

For the sum of K complex numbers z_i it follows

$$\left|\sum_{i=1}^{K} z_i\right|^2 = \left(\sum_{i=1}^{K} z_i\right)\left(\sum_{i=1}^{K} z_i\right)^* = \left(\sum_{i=1}^{K} z_i\right)\left(\sum_{i=1}^{K} z_i^*\right) = \sum_{i=1}^{K}\sum_{j=1}^{K} z_i z_j^*. \tag{23}$$

Defining the function $s(t,\phi,r,\boldsymbol{\xi};\omega)$ as

$$s(t,\phi,r,\boldsymbol{\xi};\omega) = \frac{2\sin(T\omega)}{\omega} \star \left(e^{-i\omega t} \sum_{k=-\infty}^{\infty} c_k(r) e^{ik\phi} \mathcal{F}\left[I(t-\delta t,\, \boldsymbol{\xi})\right](\omega - k\Omega(r))\right). \tag{24}$$

According to (23) we can write

$$\begin{aligned}\left|\mathcal{F}_T[f(t)](\omega)\right|^2 &= \left|\sum_j s(t_j,\, \phi_j,\, r_j,\, \boldsymbol{\xi}_j;\, \omega)\right|^2 \\ &= \sum_j \sum_l s(t_j,\, \phi_j,\, r_j,\, \boldsymbol{\xi}_j;\, \omega)\, s^*(t_l,\, \phi_l,\, r_l,\, \boldsymbol{\xi}_l;\, \omega).\end{aligned} \tag{25}$$

Due to Campbell's theorem (6),

$$\mathrm{E}\left[\left|\mathcal{F}_T[f(t)](\omega)\right|^2\right] = \mathrm{E}\left[\sum_j \sum_l s(t_j,\, \phi_j,\, r_j,\, \boldsymbol{\xi}_j;\, \omega)\, s^*(t_l,\, \phi_l,\, r_l,\, \boldsymbol{\xi}_l;\, \omega)\right]$$

$$= \int\limits_{A\times A'} s(t,\, \phi,\, r,\, \boldsymbol{\xi};\, \omega)\, s^*(t',\, \phi',\, r',\, \boldsymbol{\xi}';\, \omega)\, m_2(t,\, \phi,\, r, \boldsymbol{\xi},\, t',\, \phi',\, r',\, \boldsymbol{\xi}')\, \mathrm{d}A\, \mathrm{d}A', \quad (26)$$

where m_2 is density of the second-order moment measure corresponding to the random point process of $(t_j,\, \phi_j,\, r_j,\, \boldsymbol{\xi}_j)$. The set A is a Cartesian product of sets,

$$A = \langle -(T+C),\, T\rangle \times \langle 0,\, 2\pi\rangle \times \langle r_{\min},\, r_{\max}\rangle \times \Xi. \quad (27)$$

Now we can perform the limit (13). It can be shown that the result is independent on the value of C. In order to obtain an explicit formula for the power spectral density we have to specify the form of $M_2(.)$. In the simplest case we assume events that are mutually independent with uniformly distributed ignition times. The process can be described as a marked point process with a Poissonian process as the ground process. The intensity and the second-order measure for the ground process are:

$$M_{\mathrm{g1}}(\mathrm{d}t) = n\mathrm{d}t, \quad (28)$$

$$M_{\mathrm{g2}}(\mathrm{d}t\, \mathrm{d}t') = \left[n^2 + n\delta(t-t')\right] \mathrm{d}t\, \mathrm{d}t', \quad (29)$$

where n is the mean rate of events. Other parameters are treated as independent marks with common distribution $G(\mathrm{d}\phi\, \mathrm{d}r\, \mathrm{d}\boldsymbol{\xi})$. The second-order measure of the process has a form

$$\begin{aligned} M_2(\mathrm{d}t\, \mathrm{d}\phi\, \mathrm{d}r\, \mathrm{d}\boldsymbol{\xi}\, \mathrm{d}t'\, \mathrm{d}\phi'\, \mathrm{d}\boldsymbol{\xi}') = {}& \big[n^2 G(\mathrm{d}\phi\, \mathrm{d}r\, \mathrm{d}\boldsymbol{\xi}) G(\mathrm{d}\phi'\, \mathrm{d}r'\, \mathrm{d}\boldsymbol{\xi}') + nG(\mathrm{d}\phi\, \mathrm{d}r\, \mathrm{d}\boldsymbol{\xi}) \\ & \times\, \delta(t-t')\delta(\phi-\phi')\delta(r-r')\delta(\boldsymbol{\xi}-\boldsymbol{\xi}')\big]\, \mathrm{d}t\mathrm{d}t'. \end{aligned} \quad (30)$$

For the power spectrum we obtain this general formula,

$$\begin{aligned} S(\omega) = 4\pi^2 n \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \int\limits_{\mathcal{K}} & c_k(r) c_l^*(r) e^{i(l-k)\phi} \mathcal{F}\left[I(t-\delta t(r,\, \phi),\, \boldsymbol{\xi})\right](\omega - k\Omega(r)) \\ & \times \mathcal{F}^*\left[I(t-\delta t(r,\, \phi),\, \boldsymbol{\xi})\right](\omega - l\Omega(r))\, G(\mathrm{d}\phi\, \mathrm{d}r\, \mathrm{d}\boldsymbol{\xi}). \end{aligned} \quad (31)$$

2.5 Introducing a relationship among spots (Hawkes process)

The assumption that the spots are mutually statistically independent seems to be a reasonable first approximation, however, the actual ignition times and spot parameters should probably depend on the history of a real system. As an example of such non-Poissonian process, we calculated the power-spectral density (PSD) for a model in which the spot ignition times are distributed according to the Hawkes [17] process.

The Hawkes process consists of two types of events. Firstly, new events are generated by Poisson process operating with the intensity λ. Secondly, an existing event with ignition time t_a can give birth to new event at time t according to Poisson process with varying intensity $\mu(t-t_a)$. So the mean number of events found at time t is

$$m(t) = \lambda + \sum_{i,\, t_i<t} \mu(t_i) = \lambda + \int \mu(t) N(\mathrm{d}t). \tag{32}$$

For a stationary process the first moment density is constant. Averaging both sides of the previous equation we find,

$$m_1 = \frac{\lambda}{1-\nu}, \quad \nu = \int\limits_{-\infty}^{\infty} \mu(t)\mathrm{d}t. \tag{33}$$

Stationarity of the process implies, that the second-order measure density can depend only on the difference of its arguments. It can be proven [14] that

$$m_{\mathrm{g}2}(t,t') = c(t-t') + m_{\mathrm{g}1}^2 + m_{\mathrm{g}1}\delta(t-t'), \tag{34}$$

where the $c(t)$ is an even function. Thus, for the corresponding marked process with independent marks we find $M_2(\mathrm{d}t\,\mathrm{d}\phi\,\mathrm{d}r\,\mathrm{d}\boldsymbol{\xi}\,\mathrm{d}t'\,\mathrm{d}\phi'\,\mathrm{d}\boldsymbol{\xi}')$:

$$\begin{aligned} M_2 = \Bigg[&\left(\frac{\lambda^2}{(1-\nu)^2} + c(t-t')\right) G(\mathrm{d}\phi\,\mathrm{d}r\,\mathrm{d}\boldsymbol{\xi}) G(\mathrm{d}\phi'\,\mathrm{d}r'\,\mathrm{d}\boldsymbol{\xi}') \\ &+ \frac{\lambda}{1-\nu} G(\mathrm{d}\phi\,\mathrm{d}r\,\mathrm{d}\boldsymbol{\xi})\delta(t-t')\delta(\phi-\phi')\delta(r-r')\delta(\boldsymbol{\xi}-\boldsymbol{\xi}')\Bigg] \mathrm{d}t\mathrm{d}t'. \end{aligned} \tag{35}$$

This second-order measure is almost identical to that of the Poissonian process (there is only one additional term associated with the function $c(t)$). The resulting PSD is

$$\begin{aligned} S(\omega) = 4\pi^2 \frac{\lambda}{1-\nu} &\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \int\limits_{\mathcal{K}} e^{i(l-k)\phi} c_k(r) c_l^*(r) \mathcal{F}\left[I(t-\delta t(r,\,\phi),\,\boldsymbol{\xi})\right](\omega - k\Omega(r)) \\ &\times \mathcal{F}^*\left[I(t-\delta t(r,\,\phi),\,\boldsymbol{\xi})\right](\omega - l\Omega(r))\, G(\mathrm{d}\phi\,\mathrm{d}r\,\mathrm{d}\boldsymbol{\xi}) \ + \ 4\pi^3 \mathcal{F}\left[c(t)\right](\omega) \\ &\times \sum_{k=-\infty}^{\infty} c_k(r) \int\limits_{\mathcal{K}} e^{-ik\phi} \mathcal{F}\left[I(t-\delta t(r,\,\phi),\,\boldsymbol{\xi}')\right](\omega - k\Omega(r))\, G(\mathrm{d}\phi\,\mathrm{d}r\,\mathrm{d}\boldsymbol{\xi}) \\ &\times \sum_{l=-\infty}^{\infty} \int\limits_{\mathcal{K}'} e^{il\phi'} c_l^*(r') \mathcal{F}^*\left[I(t-\delta t(r',\,\phi'),\,\boldsymbol{\xi})\right](\omega - l\Omega(r'))\, G(\mathrm{d}\phi'\,\mathrm{d}r'\,\mathrm{d}\boldsymbol{\xi}'). \end{aligned} \tag{36}$$

The function $c(t)$ can be calculated from the mean number of secondary events $\mu(t)$. Assuming $\mu(t) = \nu\alpha\exp(-\alpha t)\theta(t)$ we obtain

$$c(t) = \frac{\lambda\alpha\nu(1-\nu/2)}{(1-\nu)^2}\exp(-\alpha(1-\nu)|t|). \tag{37}$$

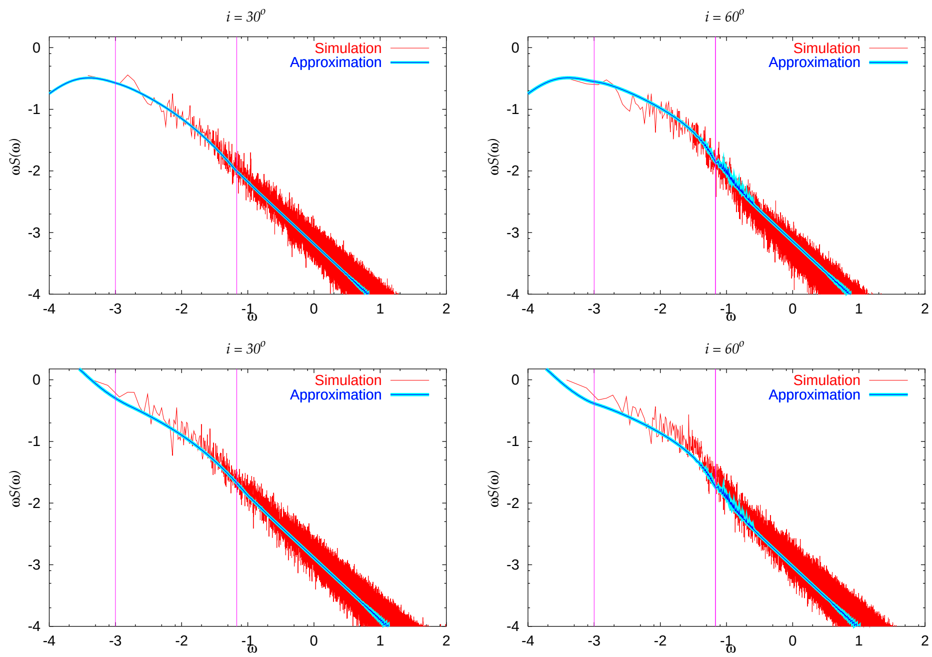

Figure 2. Power spectra from the spot model driven by the Poisson process (top row) and the Hawkes process (bottom row), calculated for spots orbiting and evolving on the surface of a thin accretion disc ($r_{\rm in} = 6$, $r_{\rm out} = 100$ gravitational radii). Two values of observer's inclination $\theta_{\rm o}$ are shown for comparison. The red (thin, noisy) curve is a result of direct numerical simulation. Blue (thick, continuous) curves are the analytical approximations based on eqs. (31) and (36), respectively. We assumed probability density function $\rho(\tau) \propto 1/\tau$. The magenta (vertical) lines denote the Keplerian orbital frequency $\Omega(r)$ at the inner and the outer edges of the disc. One can see that the Hawkes' process tends to enhance the low-frequency part of PSD and shift the break frequency towards lower values, below $\Omega(r_{\rm out})$.

It is interesting to notice that the above-given formal approach can actually provide a useful analytical formula to approximate the power spectrum. Figure 2 shows exemplary PSD which were obtained by (i) direct computations of the light-curve and the resulting PSD, and by (ii) the semi-analytical approach with Poissonian and Hawkes processes.

3 CONCLUSIONS

We have studied the properties of power spectral density within the model of accretion disc variability driven by orbiting spots. The origin and evolution of spots were described in terms of Poissonian and Hawkes' processes. The latter belongs to

a category of avalanche models. We developed an analytical approximation of PSD and compared it with our numerical results from light-curve simulations. In this way we were able to demonstrate the precision of formulae (31) and (36). The analytical approximation evaluates very fast and provides the main trend of the PSD shape while avoiding the noisy form of the numerically simulated spectra. Our approach allows us to investigate the resulting PSD as a function of the assumed type of process, which describes creation of parent spots and the subsequent cascades of daughter spots. In particular, we can investigate the predicted PSD slope at different frequency ranges and we can locate the break frequency depending on the model parameters.

The resulting PSD can be approximated by a broken power-law. For every stationary process the quantities $S(0)$ and $\int_0^\infty S(\omega)\mathrm{d}\omega$ are finite. Therefore, the function $S(\omega)$ flattens ($S(\omega) \approx \omega^0$) near $\omega = 0$ and it must decrease faster than $1/\omega$ at high frequencies. Power-spectra generated by the spot model behave in this way. The low-frequency limit is a constant, whereas the-high frequency behaviour depends mainly on the shape of the spot emission profile, $I(t,\, \boldsymbol{\xi}_j)$. In our calculations the emissivity was a decaying exponential and the slope was equal to -2 at high frequencies. The most interesting part of the spectra in between those two limits is influenced by both the emissivity profile and the underlying process.

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